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The Maxwell operator on q-Minkowski space and q-hyperboloid algebras

Antoine Dutriaux and Dimitri Gurevich

LAMAV, Université de Valenciennes, 59304 Valenciennes, France

E-mail: dutriaux@univ-valenciennes.fr and gurevich@univ-valenciennes.fr

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Abstract
We introduce an analog of the Maxwell operator on a q-Minkowski space algebra (treated as a particular case of the so-called reflection equation algebra) and on certain of its quotients. We treat the space of 'quantum differential forms' as a projective module in the spirit of the Serre–Swan approach. Also, we use 'braided tangent vector fields' which are q-analogs of Poisson vector fields associated with the Lie bracket $sl(2)$.

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1. Introduction

The main goal of this paper is to define a q-analog of the Maxwell operator on some noncommutative (NC) algebras. Namely, we are dealing with three of them: the q-Minkowski space algebra $\mathbb{K}_q[\mathbb{R}^4]$, quantum (braided, or q-)hyperboloid algebra $\mathbb{K}_q\big[\mathrm{H}_r^2\big]$ and an intermediate algebra $\mathbb{K}_q[\mathbb{R}^3]$. The q-Minkowski space algebra (as defined in [MM, M, K]) is a particular case of a so-called reflection equation algebra (REA), the others are its quotients. Observe that the REA was introduced in the early 1990s by S Majid under the name of braided matrix algebra (cf the cited papers and the references therein). He also defined a Hermitian structure in it. Here we do not consider such a structure[1].

The above algebras are deformations of their classical counterparts $\mathbb{K}[\mathbb{R}^4]$, $\mathbb{K}\big[\mathrm{H}_r^2\big]$ and $\mathbb{K}[\mathbb{R}^3]$, respectively. Hereafter $\mathbb{K} = \mathbb{C}$ (or $\mathbb{R}$) is the ground field, the notation $\mathbb{K}[\mathcal{M}]$ stands for the coordinate algebra of a given regular affine algebraic variety $\mathcal{M}$, and

$$\mathrm{H}_r^2 = \left\{(b, h, c) \in \mathbb{R}^3 \,\middle|\, 2bc + \frac{h^2}{2} = r^2, r \neq 0\right\} \tag{1.1}$$

[1] The problem of defining involution operators in 'braided algebras' will be discussed in subsequent papers. (Hereafter the term 'braided' stands for the REA and related algebras and objects.) We would like only to note that the q-Minkowski space algebra endowed with the mentioned Hermitian structure cannot be treated as a real vector space.

is a hyperboloid[2]. Moreover, all deformed algebras we deal with can be endowed with an action of the quantum group (QG) $U_q(sl(2))$ compatible with their product in the usual way.

By passing from the classical algebras to their quantum or braided analogs we want to simultaneously deform certain differential operators defined on the initial algebras or some vector bundles over them. Moreover, if a given operator is covariant with respect to a group G we require its deformed counterpart to be covariant with respect to the corresponding QG. The simplest operator which can be 'q-deformed' is the Laplace (or Laplace–Beltrami) operator on one of the mentioned algebras. Recall that this operator is associated with a (pseudo-)Riemannian metric on a given regular variety. Thus, if the metric g which comes in its definition is constant of the signature $(1|3)$ on the space $\mathbb{R}^4$ the corresponding Laplace operator (also called the d'Alembertian) is $\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2$ where (t, x, y, z) are the Cartesian coordinates in this space. If $\mathcal{M} = \mathrm{H}_r^2$ and $g \in \Omega^2(\mathrm{H}_r^2)$ is an $SL(2)$-invariant metric the corresponding Laplace operator is equal (up to a factor) to the quadratic Casimir operator coming from the enveloping algebra $U(sl(2))$ whereas the hyperboloid is treated as an orbit $\mathrm{H}_r^2 \hookrightarrow sl(2)^*$ of the coadjoint action of the Lie algebra $sl(2)$. Vector fields arising from this coadjoint action are tangent to all orbits in $sl(2)^*$ and we call them *tangent vector fields*. (Also, they are Poisson vector fields with respect to the linear Poisson–Lie bracket defined on the space $sl(2)^*$.)

Thus, in order to define braided analogs of the Laplace operator on the quantum algebras in question we should first introduce braided analogs of vector fields. More precisely, we need analogs of tangent vector fields while dealing with the algebra $\mathbb{K}_q[\mathrm{H}_r^2]$, and analogs of partial derivatives when dealing with the algebras $\mathbb{K}_q[\mathbb{R}^3]$ and $\mathbb{K}_q[\mathbb{R}^4]$. The problem of defining a braided analog of the Maxwell operator is even more complicated since we should first introduce braided analogs of the spaces of differential forms $\Omega^1(\mathrm{H}_r^2)$, $\Omega^1(\mathbb{R}^3)$ and $\Omega^1(\mathbb{R}^4)$. Recall that the Maxwell operator is defined on a given variety $\mathcal{M}$ as follows:

$$\omega \to \partial d\omega, \qquad \partial = *^{-1} d\, *. \tag{1.2}$$

Here $\omega \in \Omega^1(\mathcal{M})$ is a one-form on $\mathcal{M}$, d is the de Rham operator and $*$ is the Hodge operator. (Note that on the classical Minkowski space the Maxwell system is initially defined on the space $\Omega^2(\mathbb{R}^4)$ but it can be easily reduced to the operator above. Also, note that the conventional definitions of the Maxwell and Laplace operators differ from ours by a sign. We disregard this.)

There are several approaches to the problem of defining analogs of differential forms and of the de Rham operator on a given noncommutative algebra A. The first approach consists of considering *universal differential forms* without any commutation relations (e.g., $a\, db = db\, a$) between 'functions' $a \in A$ and 'differentials' db but with the preservation of the Leibniz rule. The corresponding differential algebra is much bigger than the algebra of usual differential forms even if the initial algebra is the coordinate algebra $A = \mathbb{K}[\mathcal{M}]$ of a regular variety $\mathcal{M}$.

If an algebra A is related to a braiding (say, it is a so-called RTT algebra or an REA, see section 4) one looks for an extension of the braiding coming in the definition of A onto the space of differential forms. Such an extended braiding enables one to relate the elements of the form $a\,\mathrm{d}b$ and $\mathrm{d}b\, a$, and to reduce the space of universal differential forms to the 'classical size'. In the case of the quantum analog of the group $GL(n)$ this can even be done with preservation of the Leibniz rule (cf [W, K, IP]) while for the quantum analog of the group $SL(n)$ this rule has to be dropped (cf [FP]).

[2] If $\mathbb{K} = \mathbb{R}$ and r is real, we get a one-sheeted hyperboloid. If r is purely imaginary, we get a two-sheeted hyperboloid. However, if $\mathbb{K} = \mathbb{C}$ we allow r to take any non-trivial value. It should be emphasized that we prefer to deal with a q-analog of a hyperboloid (and not of a sphere—the so-called Podles' sphere) since it cannot be realized as a real algebra.

The third approach, due to A Connes, is based on the notion of spectral triples. In the framework of this approach the role of differential forms is played by the (classes of) Hochschild cycles [C].

Nevertheless, all these approaches do not enable one to define a smoothly deformed space of differential forms on a quantum hyperboloid algebra $\mathbb{K}_q\left[\mathrm{H}_r^2\right]$. As was observed in [AG] the space of differential forms $\Omega^1\left(\mathrm{H}_r^2\right)$ can be smoothly deformed into a quantum one $\Omega_q^1\left(\mathrm{H}_r^2\right)$ as a one-sided module. However, opposite to the classical case ($q = 1$), if we convert this one-sided $\mathbb{K}_q\left[\mathrm{H}_r^2\right]$-module into a two-sided module via an extension of the initial braiding we reduce the size of the space of differential forms.

Following [AG, A], we treat the spaces of braided differential forms as one-sided projective modules[3]. In order to do so, we use the following remarkable property of the q-Minkowski space algebra $\mathbb{K}_q[\mathbb{R}^4]$. There exists a series of the Cayley–Hamilton (CH) polynomial identities for some matrices with entries from the algebra $\mathbb{K}_q[\mathbb{R}^4]$. The coefficients of the CH polynomials are central in this algebra, and they become scalar if we switch to the quotient $\mathbb{K}_q\left[\mathrm{H}_r^2\right]$. It is a somewhat standard trick to use these polynomials to construct a set of idempotents and the corresponding projective modules. Thus, we can explicitly deform any $SL(2)$-equivariant vector bundle on a hyperboloid H_r^2, realized as a projective module, to its braided counterpart. By applying this scheme to the space $\Omega^1\left(\mathrm{H}_r^2\right)$ we get its braided analog $\Omega_q^1\left(\mathrm{H}_r^2\right)$, realized as a projective $\mathbb{K}_q\left[\mathrm{H}_r^2\right]$-module. Moreover, it is $U_q(sl(2))$-equivariant (covariant).

Besides, we define braided analogs of tangent vector fields without any form of the Leibnitz rule. In order to do that, we use another remarkable property of the q-Minkowski space algebra $\mathbb{K}_q[\mathbb{R}^4]$. Let $\mathcal{L}$ be the space spanned by the generators of this algebra. There exists a braided analog

$$[\ ,\ ]_q : \mathcal{L}^{\otimes 2} \to \mathcal{L}$$

of the $gl(2)$ Lie bracket such that (a slight modification of) the algebra $\mathbb{K}_q[\mathbb{R}^4]$ can be regarded as the enveloping algebra of the corresponding q- (or braided) Lie algebra. We refer the reader to [GPS2] for further explanations (applicable in a much more general setting). Below, we only need a braided analog $[\ ,\ ]_q : \mathcal{SL}^{\otimes 2} \to \mathcal{SL}$ of the Lie algebra $sl(2)$ where $\mathcal{SL}$ is a three-dimensional subspace of the space $\mathcal{L}$. By using this q-bracket we define 'braided tangent vector fields' following the classical pattern. Moreover, there is a braided analog of the quadratic Casimir element on the space $\mathcal{SL}^{\otimes 2}$. Representing it by the above 'braided tangent vector fields' we get an analog of the Casimir operator on a q-hyperboloid.

A braided analog of the Maxwell operator on a q-hyperboloid is also defined via braided tangent vector fields as a proper deformation of the Maxwell operator acting on $\Omega^1\left(\mathrm{H}_r^2\right)$. However, as we have said above, the space of braided differential forms on a q-hyperboloid is treated as a one-sided projective $\mathbb{K}_q\left[\mathrm{H}_r^2\right]$ module. In order to do this we first apply this scheme to the Maxwell operator on a usual hyperboloid (which seems to be new even in this classical setting).

In order to get the Maxwell operator on the algebra $\mathbb{K}_q[\mathbb{R}^3]$, in addition to 'braided tangent vector fields', we must also use the derivative in r where r is an analog of the radius (it comes in a parametrization of quantum hyperboloids and in a sense belongs to the algebraic extension of the center of the algebra $\mathbb{K}_q[\mathbb{R}^3]$). We assume that this derivative has the classical

[3] Note that in general, due to the Serre–Swan approach, any vector bundle on a regular affine variety can be realized as a projective module. Recall that such (say, right) A-module for a given algebra A is of the form $eA^{\oplus n}$ where $e \in \mathrm{Mat}_n(A)$ is an idempotent and $\mathrm{Mat}_n(A)$ stands for the space of $n \times n$ matrices with entries from A. As was shown in [R], if $A_\hbar$ is a formal deformation of a commutative algebra A, any idempotent $e \in \mathrm{Mat}_n(A)$ can be deformed into an idempotent $e_\hbar \in \mathrm{Mat}_n(A_\hbar)$. Thus, we get a one-sided projective $A_\hbar$-module $e_\hbar A_\hbar^{\oplus n}$ which is a formal deformation of the initial A-module. (Nowadays, there is a known explicit formula for such a deformed idempotent, cf [BB].)

properties, in particular, it satisfies the Leibnitz rule. Thus, we relate the braided Laplace and Maxwell operators on the algebra $\mathbb{K}_q\left[\mathrm{H}_r^2\right]$ and on the algebra $\mathbb{K}_q[\mathbb{R}^3]$ in a way similar to the classical one: the operators on the former algebra are restrictions of those on the latter one. Emphasize that the methods of defining partial derivatives on the q-Minkowski space algebra via the 'braided Leibnitz rule' (based on a transposition of 'functions' and 'partial derivatives' as in [K, IP, FP] or a braided coaddition as in [M]) do not allow one to get braided vector fields on a q-hyperboloid. Also, note that our braided vector fields differ drastically from q-analogs of differential operators arising from a coalgebraic structure in the corresponding QG (cf [D]).

As for the q-Minkowski space algebra $\mathbb{K}_q[\mathbb{R}^4]$, it has one generator more compared to $\mathbb{K}_q[\mathbb{R}^3]$. Moreover, this generator is central. So, first we define the partial derivative with respect to this generator, and then we introduce the Maxwell operators on the algebra $\mathbb{K}_q[\mathbb{R}^4]$ in the classical manner.

By properly defining the action of the QG $U_q(sl(2))$ on all ingredients of the Maxwell operators on the algebras in question, we force them to be $U_q(sl(2))$-invariant. Moreover, these operators possess gauge freedom similar to the classical one (i.e. their kernels are as large as the kernels of their classical counterparts), provided the corresponding Laplace operators are central.

In conclusion we want to mention the paper [S] where the author suggests a way of defining q-analogs of gauge models via quantum gauge potentials. For this end he uses a q-analog of the Lie algebra $su(n)$ similar to that considered above. However, the ground (source) algebra considered in [S] is commutative whereas our ground algebras are essentially noncommutative.

We hope our method will be useful for a 'q-deformation' of other gauge models.

2. Maxwell operator via projective modules

In this section we introduce the Maxwell operator in classical settings and consider its behavior with respect to the restriction to a subvariety. Also, we give a few basic examples.

Let $\mathcal{M}$ be a regular affine variety endowed with a (pseudo-) Riemannian metric $g_{ij} = g(\partial_i, \partial_j)$ where ∂_i are partial derivatives in local coordinates. We need two operators

$$d : \mathbb{K}[\mathcal{M}] \to \Omega^1(\mathcal{M}), \qquad f(x) \rightarrowtail \partial_i f\, \mathrm{d}x_i,$$

$$\partial : \Omega^1(\mathcal{M}) \to \mathbb{K}[\mathcal{M}], \qquad \alpha_i\, \mathrm{d}x_i \rightarrowtail \frac{1}{\sqrt{g}}\partial_i(g^{ij}\sqrt{g}\alpha_j),$$

where $g = |\det(g_{ij})|$ and the tensor g^{ij} is inverse to g_{ij}. The Laplace operator on the algebra $\mathbb{K}[\mathcal{M}]$ is

$$\Delta(f) = \partial d f = \frac{1}{\sqrt{g}}\partial_i(g^{ij}\sqrt{g}\partial_j f).$$

The Laplace operators on the spaces $\Omega^i(\mathcal{M})$ are defined by the formula

$$\Delta = \partial d + d\partial,$$

where $\partial : \Omega^i(\mathcal{M}) \to \Omega^{i-1}(\mathcal{M})$ is the well-known analog of the above operator. In what follows we realize the Maxwell operator as $\mathrm{Mw} = \Delta - d\partial$. Besides, if $\mathcal{M} = \mathbb{R}^n$ and the metric is constant in the Cartesian coordinates x_i, the operator Δ acts on the space $\Omega^1(\mathcal{M})$ via $\Delta(\alpha_i\, \mathrm{d}x_i) = \Delta(\alpha_i)\, \mathrm{d}x_i$.

Proposition 1. *Let $\mathcal{N} \subset \mathcal{M}$ be a regular subvariety of a variety $\mathcal{M}$ of codimension 1. Suppose that in a vicinity of each point $a \in \mathcal{N}$ there exists a coordinate system $(x_1, x_2, \ldots, x_n)$ such that*

$\mathcal{N}$ is given by $x_n = 0$ and $(x_1, x_2, \ldots, x_{n-1})$ is a local coordinate system in $\mathcal{N}$, $g(\partial_n, \partial_n) = 1$ and $g(\partial_i, \partial_n) = 0$ for any $1 \leqslant i \leqslant n-1$. Then $\mathrm{Mw}_{\mathcal{N}} = \mathrm{Mw}_{\mathcal{M}}|_{\mathcal{N}}$, i.e. the Maxwell operator on $\mathcal{N}$ is the restriction of the Maxwell operator on $\mathcal{M}$. A similar claim is valid for the Laplace operator.

Proof. Proof It follows from the explicit form of the Maxwell and Laplace operators in the local coordinate system $(x_1, x_2, \ldots, x_n)$. □

This proposition enables us to write the Maxwell operator on certain algebraic varieties in terms of ambient spaces. Thus, we realize the space of one-forms (as well as the space of vector fields) on such a variety as a projective module without using any local coordinate system.

Let us consider the basic example–a sphere

$$S_r^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = r^2, r > 0\}$$

embedded in the Euclidean space $\mathbb{R}^3 \cong so(3)^*$ as an orbit of action of the group $SO(3)$. Also, we assume this space to be equipped with an $SO(3)$-invariant pairing $\langle x, x\rangle = 1$, $\langle x, y\rangle = 0$, and so on. The corresponding metric is $g(\partial_x, \partial_x) = 1$, $g(\partial_x, \partial_y) = 0$, and so on. Also, we endow the coordinate algebra $\mathbb{K}[\mathbb{R}^3]$ of the space $\mathbb{R}^3$ with the $SO(3)$-covariant Poisson bracket

$$\{x, y\} = z, \qquad \{y, z\} = x, \qquad \{z, x\} = y.$$

To any function $f \in \mathbb{K}[\mathbb{R}^3]$ we associate the operator

$$\mathrm{Pois}_f(g) := \{f, g\}, \qquad \forall g \in \mathbb{K}[\mathbb{R}^3].$$

Then the operators $X = \mathrm{Pois}_x$, $Y = \mathrm{Pois}_y$, $Z = \mathrm{Pois}_z$ are infinitesimal rotations. Their explicit forms are

$$X = z\partial_y - y\partial_z, \qquad Y = x\partial_z - z\partial_x, \qquad Z = y\partial_x - x\partial_y.$$

They are tangent to the spheres S_r^2 and subject to the relation

$$xX + yY + zZ = 0.$$

Consider the coordinate algebra of the sphere S_r^2

$$\mathbb{K}[S_r^2] = \mathbb{K}[\mathbb{R}^3]/\langle x^2 + y^2 + z^2 - r^2\rangle.$$

Hereafter $\langle I\rangle$ stands for the two-sided ideal generated by a set I. The space $\mathrm{Vect}(S_r^2)$ of all vector fields on a sphere (with coefficients from $\mathbb{K}[S_r^2]$), treated as a $\mathbb{K}[S_r^2]$-module, is the quotient

$$M = \mathbb{K}[S_r^2]^{\oplus 3}/\overline{M}$$

of the free $\mathbb{K}[S_r^2]$-module $\mathbb{K}[S_r^2]^{\oplus 3}$ over the submodule $\overline{M} = \{\varphi(xX + yY + zZ), \forall\varphi \in \mathbb{K}[S_r^2]\}$.

It is not difficult to see that the module $\overline{M}$ is projective. Indeed, the matrix

$$\overline{e} = \frac{1}{r^2}\begin{pmatrix} x \\ y \\ z \end{pmatrix}(x \quad y \quad z)$$

defines an idempotent such that $\overline{M} = \overline{e}\mathbb{K}[S_r^2]^{\oplus 3}$. Therefore the $\mathbb{K}[S_r^2]$-module M can be realized as a submodule

$$M = e\mathbb{K}[S_r^2]^{\oplus 3} \subset \mathbb{K}[S_r^2]^{\oplus 3},$$

where $e = I - \overline{e}$ is the complementary idempotent.

We call the $\mathbb{K}[S_r^2]$-module M *tangent*. In contrast with other $\mathbb{K}[S_r^2]$-modules, the tangent module defines the action $M \otimes \mathbb{K}[S_r^2] \to \mathbb{K}[S_r^2]$ consisting of applying a vector field to a function.

In a similar way we can realize the space of one-forms $\Omega^1(S_r^2)$. This space consists of all elements $\{\alpha\,\mathrm{d}x+\beta\,\mathrm{d}y+\gamma\,\mathrm{d}z\}$ whereas the elements $\overline{M} = \{\varphi(x\,\mathrm{d}x+y\,\mathrm{d}y+z\,\mathrm{d}z), \forall\varphi \in \mathbb{K}[S_r^2]\}$ vanish. Thus, as a $\mathbb{K}[S_r^2]$-module, the space $\Omega^1(S_r^2)$ can be treated as the former module. The latter module is called *cotangent*. (For a similar treatment of the space $\Omega^2[S_r^2]$ the reader is referred to [GS1].)

We do not distinguish between the tangent and cotangent $\mathbb{K}[S_r^2]$-modules, and denote them $M(S_r^2)$. Its elements are treated to be triples $(\alpha, \beta, \gamma)^t$ (t stands for transposing) modulo

$$(x\rho, y\rho, z\rho)^t, \qquad \alpha, \beta, \gamma, \rho \in \mathbb{K}[S_r^2].$$

Now, we exhibit the Maxwell operator on the Euclidian space $\mathbb{R}^3$ in a convenient form. This operator acts on the space of one-differential forms

$$\Omega^1(\mathbb{R}^3) = \{\alpha\,\mathrm{d}x + \beta\,\mathrm{d}y + \gamma\,\mathrm{d}z\}, \qquad \alpha, \beta, \gamma \in \mathbb{K}[\mathbb{R}^3]$$

(which is a free $\mathbb{K}[\mathbb{R}^3]$-module $\Omega^1(\mathbb{R}^3) \cong \mathbb{K}[\mathbb{R}^3]^{\oplus 3}$) via formula (1.2).

It can also be written in the following form

$$-\mathrm{rot\,rot} = \Delta - \mathrm{grad\,div},$$

where

$$\mathrm{rot} : \Omega^1(\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3), \qquad \mathrm{div} : \Omega^1(\mathbb{R}^3) \to \mathbb{K}[\mathbb{R}^3], \qquad \mathrm{grad} : \mathbb{K}[\mathbb{R}^3] \to \Omega^1(\mathbb{R}^3)$$

are the curl, divergence and gradient respectively and

$$\Delta = \Delta_{\mathbb{K}[so(3)^*]} = \partial_x^2 + \partial_y^2 + \partial_z^2$$

is the Laplace operator.

By identifying a differential form $\alpha\,\mathrm{d}x+\beta\,\mathrm{d}y+\gamma\,\mathrm{d}z$ and the triple $(\alpha, \beta, \gamma)^t$ as explained above we can write

$$\mathrm{Mw}_{\mathbb{K}[so(3)^*]}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \begin{pmatrix}\Delta(\alpha)\\ \Delta(\beta)\\ \Delta(\gamma)\end{pmatrix} - \begin{pmatrix}\partial_x\\ \partial_y\\ \partial_z\end{pmatrix}(\partial_x, \partial_y, \partial_z)\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}. \tag{2.1}$$

Observe that the Maxwell operator is $SO(3)$-invariant if $SO(3)$ acts on the generators x, y, z and the matrices $M \in \mathrm{Mat}_3(\mathbb{R})$ in a proper way (see section 5). The gauge freedom is due to the fact that the triples $(\partial_x\rho, \partial_y\rho, \partial_z\rho)^t$ belong to the kernel $\mathrm{Ker}(\mathrm{Mw}_{\mathbb{K}[\mathrm{so}(3)^*]})$ of this operator. So, if a triple $(\alpha, \beta, \gamma)^t$ is a solution of the Maxwell equation $\mathrm{Mw}_{\mathbb{K}[so(3)^*]}(\alpha, \beta, \gamma)^t = (\lambda, \mu, \nu)^t$ then the triple $(\alpha, \beta, \gamma)^t + (\partial_x\rho, \partial_y\rho, \partial_z\rho)^t$ is also a solution.

Now, consider the Maxwell operator on the sphere S_r^2. By using the relation

$$r\partial_r = x\partial_x + y\partial_y + z\partial_z \tag{2.2}$$

we realize the partial derivatives $\partial_x, \partial_y, \partial_z$ as follows:

$$\partial_x = \frac{1}{r^2}(yZ - zY) + \frac{x}{r}\partial_r \circlearrowleft. \tag{2.3}$$

(Thereafter the symbol $\circlearrowleft$ stands for the cyclic permutations). In what follows we also use the following formulae

$$\partial_r x = \frac{x}{r} \circlearrowleft, \qquad \partial_x r = \frac{x}{r} \circlearrowleft, \qquad X(r) = Y(r) = Z(r) = 0,$$

and the fact that the vector field ∂_r commutes with X, Y, Z.

By using formula (2.3) we rewrite the Laplace operator on the space $\mathbb{R}^3$ as follows:

$$\Delta_{\mathbb{K}[so(3)^*]} = \left(\frac{1}{r^2}\mathcal{X} + \frac{x}{r}\partial_r\right)^2 + \left(\frac{1}{r^2}\mathcal{Y} + \frac{y}{r}\partial_r\right)^2 + \left(\frac{1}{r^2}\mathcal{Z} + \frac{z}{r}\partial_r\right)^2, \tag{2.4}$$

where we use the notations

$$\mathcal{X} = yZ - zY, \qquad \mathcal{Y} = zX - xZ, \qquad \mathcal{Z} = xY - yX.$$

By the above proposition the Laplacian $\Delta_{\mathbb{K}[S_r^2]}$ on the sphere S_r^2 is the restriction of the Laplacian $\Delta_{\mathbb{K}[so(3)^*]}$ to the sphere S_r^2. Indeed, in spherical coordinates the radius r plays the role of the coordinate x_n from proposition 1.

Lemma 2.

$$\Delta_{\mathbb{K}[S_r^2]} = \frac{\mathcal{X}^2 + \mathcal{Y}^2 + \mathcal{Z}^2}{r^4}. \tag{2.5}$$

Proof. We have only to check that the first order component of the operator (2.4) vanishes on the sphere S_r^2 $\partial_r = 0$. Indeed, this component equals

$$\frac{x}{r}\partial_r\left(\frac{1}{r^2}\right)\mathcal{X} + \frac{y}{r}\partial_r\left(\frac{1}{r^2}\right)\mathcal{Y} + \frac{z}{r}\partial_r\left(\frac{1}{r^2}\right)\mathcal{Z} = -\frac{2}{r^4}(x\mathcal{X} + y\mathcal{Y} + z\mathcal{Z}) = 0.$$

(Moreover, there are no $SO(3)$-invariant first order differential operators on the space $\mathbb{R}^3$.) □

Lemma 3. *The Laplace operator (2.5) equals*

$$\Delta_{\mathbb{K}[S_r^2]} = \frac{X^2 + Y^2 + Z^2}{r^2}.$$

Proof. We have

$$\begin{aligned}
\mathcal{X}^2 + \mathcal{Y}^2 + \mathcal{Z}^2 &= (yZ - zY)^2 + (zX - xZ)^2 + (xY - yX)^2 \\
&= y^2Z^2 + z^2Y^2 - yz(YZ + ZY) - yxZ + zxY + \circlearrowleft \\
&= (r^2 - x^2)X^2 - yxYX - zxZX + \circlearrowleft \\
&= r^2(X^2 + Y^2 + Z^2) - [x(xX + yY + zZ)X + \circlearrowleft] = r^2(X^2 + Y^2 + Z^2).
\end{aligned}$$ □

This form of the Laplacian is more familiar and widely used in the study of rotationally symmetric Schrodinger operators.

Let us emphasize that the operators $X, Y, Z, \mathcal{X}, \mathcal{Y}, Z$ are well defined on the space $\mathbb{R}^3$, and the relation $\mathcal{X}^2 + \mathcal{Y}^2 + \mathcal{Z}^2 = r^2(X^2 + Y^2 + Z^2)$ is also valid on the whole space $\mathbb{R}^3$.

Proposition 4.

$$\Delta_{\mathbb{K}[S_r^2]}\mathcal{X} - \mathcal{X}\Delta_{\mathbb{K}[S_r^2]} = -2x\Delta_{\mathbb{K}[S_r^2]} \circlearrowleft$$

on $\mathbb{R}^3$.

Proof. Indeed, by direct computations we have

$$\begin{aligned}
\Delta_{\mathbb{K}[S_r^2]}\mathcal{X} - \mathcal{X}\Delta_{\mathbb{K}[S_r^2]} &= \frac{2}{r^2}(zZX + yYX - xY^2 - xZ^2) \\
&= \frac{2}{r^2}((xX + yY + zZ)X - x(X^2 + Y^2 + Z^2)).
\end{aligned}$$ □

Now, define the Maxwell operator $\mathrm{Mw}_{\mathbb{K}[S_r^2]}$ on the sphere S_r^2 as follows:

$$\mathrm{Mw}_{\mathbb{K}[S_r^2]}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = e\left(\begin{pmatrix}\Delta_{\mathbb{K}[S_r^2]}(\alpha)\\ \Delta_{\mathbb{K}[S_r^2]}(\beta)\\ \Delta_{\mathbb{K}[S_r^2]}(\gamma)\end{pmatrix} - \frac{1}{r^4}\begin{pmatrix}\mathcal{X}\\ \mathcal{Y}\\ \mathcal{Z}\end{pmatrix}(\mathcal{X},\mathcal{Y},\mathcal{Z})\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}\right), \qquad \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}\in M\left(S_r^2\right). \tag{2.6}$$

In this definition we assume that the elements of the module $M\left(S_r^2\right)$ are triples $(\alpha,\beta,\gamma)^t$ such that $\bar{e}(\alpha,\beta,\gamma)^t = 0$ or, which is the same, $e(\alpha,\beta,\gamma)^t = (\alpha,\beta,\gamma)^t$. The idempotent e coming in this definition ensures that the image of the operator $\mathrm{Mw}_{\mathbb{K}[S_r^2]}$ belongs to the module $M\left(S_r^2\right)$.

Now, we justify this definition. In fact, by proposition 1 if we present the Maxwell operator on $\mathbb{R}^3$ in spherical coordinates and restrict it to the sphere S_r^2, we get the Maxwell operator on this sphere. It remains to check that it coincides with the operator (2.6).

Let us observe that similar to the Laplacian (2.5) the operator $\mathrm{Mw}_{\mathbb{K}[S_r^2]}$ is $SO(3)$-covariant. Moreover, its gauge freedom consists of the fact that the triples $(\mathcal{X}(\rho),\mathcal{Y}(\rho),\mathcal{Z}(\rho))^t$ belong to $\mathrm{Ker}(\mathrm{Mw}_{\mathbb{K}[S_r^2]})$. It follows from proposition 4. (These triples belong to the module $M\left(S_r^2\right)$ since $\bar{e}(\mathcal{X},\mathcal{Y},\mathcal{Z})^t = 0$.)

Remark 5. Introduce 'gradient' and 'divergence' on the sphere S_r^2 as follows:

$$\mathrm{grad}_{\mathbb{K}[S_r^2]} f = r^{-2}(\mathcal{X}(f),\mathcal{Y}(f),\mathcal{Z}(f)),$$

$$\mathrm{div}_{\mathbb{K}[S_r^2]}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = r^{-2}(\mathcal{X},\mathcal{Y},\mathcal{Z})\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}, \qquad \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}\in M\left(S_r^2\right).$$

Rewrite the operator $\mathrm{Mw}_{\mathbb{K}[S_r^2]}$ in the form similar to the classical one:

$$\mathrm{Mw}_{\mathbb{K}[S_r^2]} = e\Delta_{\mathbb{K}[S_r^2]} - \mathrm{grad}_{\mathbb{K}[S_r^2]}\,\mathrm{div}_{\mathbb{K}[S_r^2]}$$

(the factor e in the second summand can be omitted).

Let us consider one more example–the classical Minkowski space, i.e. a four-dimensional space endowed with an $SO(1,3)$-covariant norm

$$\|(t,x,y,z)\| = \sqrt{t^2-x^2-y^2-z^2}.$$

The corresponding second-order differential operator is

$$\Delta_{\mathbb{K}[\mathbb{R}^4]} = \partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2.$$

It is called the d'Alembertian or the $so(1,3)$ Laplacian. Then the corresponding Maxwell operator is

$$\mathrm{Mw}_{\mathbb{K}[\mathbb{R}^4]}\begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{pmatrix} = \begin{pmatrix}\Delta_{\mathbb{K}[\mathbb{R}^4]}(\alpha)\\ \Delta_{\mathbb{K}[\mathbb{R}^4]}(\beta)\\ \Delta_{\mathbb{K}[\mathbb{R}^4]}(\gamma)\\ \Delta_{\mathbb{K}[\mathbb{R}^4]}(\delta)\end{pmatrix} - \begin{pmatrix}\partial_t\\ \partial_x\\ \partial_y\\ \partial_z\end{pmatrix}(\partial_t - \partial_x, -\partial_y, -\partial_z)\begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{pmatrix}. \tag{2.7}$$

It is evident that it is $SO(1,3)$-covariant and

$$(\partial_t\rho,\partial_x\rho,\partial_y\rho,\partial_z\rho)^t \in \mathrm{Ker}(\mathrm{Mw}).$$

It is also clear that by restricting $t=0$ we get the Maxwell operator on the space $\mathbb{R}^3$ (up to a sign).

3. Maxwell operator on $sl(2)^*$ and H_r^2

Now, apply the above scheme to the space $\mathbb{R}^3 \cong sl(2)^*$ endowed with an action of the group $SL(2)$. This example is another (non-compact) real form of the situation considered above. So, the corresponding Maxwell operator can be obtained by a mere change of a basis. Nevertheless, we describe it in detail since it is going to be 'q-deformed' below.

Consider a basis $\{b, h, c\}$ in the space $\mathbb{K}[sl(2)^*]$ equipped with the $SL(2)$-covariant Poisson bracket

$$\{h, b\} = 2b, \qquad \{h, c\} = -2c, \qquad \{b, c\} = h.$$

The corresponding Poisson operators are

$$H = \mathrm{Pois}_h = 2b\partial_b - 2c\partial_c, \qquad B = \mathrm{Pois}_b = h\partial_c - 2b\partial_h, \qquad C = \mathrm{Pois}_c = -h\partial_b + 2c\partial_h.$$

They are tangent to hyperboloids and subject to the relation

$$cB + \frac{hH}{2} + bC = 0. \tag{3.1}$$

The $\mathbb{K}\left[\mathrm{H}_r^2\right]$-module $\mathrm{Vect}\left(\mathrm{H}_r^2\right)$ of vector fields on a hyperboloid (1.1) is a quotient module $M = \mathbb{K}\left[\mathrm{H}_r^2\right]^{\oplus 3}/\overline{M}$ of the free $\mathbb{K}\left[\mathrm{H}_r^2\right]$-module $\mathbb{K}\left[\mathrm{H}_r^2\right]^{\oplus 3}$ over the submodule $\overline{M} = \left\{\varphi(cB + \frac{hH}{2} + bC), \forall \varphi \in \mathbb{K}\left[\mathrm{H}_r^2\right]\right\}$. The idempotent corresponding to the module $\overline{M}$ is

$$\overline{e} = \frac{1}{r^2}\begin{pmatrix} c \\ \frac{h}{2} \\ b \end{pmatrix}(b \quad h \quad c) \tag{3.2}$$

(In order to show that $\overline{M} = \overline{e}\mathbb{K}^{\oplus 3}$ it suffices to check that $\mathbb{K}\left[\mathrm{H}_r^2\right] = \{b\alpha + h\beta + c\gamma\}$.) Thus, the module M is also projective $M = e\mathbb{K}\left[\mathrm{H}_r^2\right]^{\oplus 3}$ where $e = 1 - \overline{e}$. The $\mathbb{K}\left[\mathrm{H}_r^2\right]$-module $\Omega^1\left(\mathrm{H}_r^2\right)$ can be treated similarly.

Endow the space $\mathrm{span}(b, h, c)$ with an $SL(2)$-covariant pairing

$$\langle b, c\rangle = \langle c, b\rangle = 1, \qquad \langle h, h\rangle = 2 \tag{3.3}$$

which is inverse to the Casimir element

$$\mathrm{Cas} = bc + \frac{h^2}{2} + cb.$$

Thus, on the space $sl(2)$

$$\langle b = \partial_c, \qquad \langle c = \partial_b, \qquad \langle h = 2\partial_h, \tag{3.4}$$

where $\langle x : sl(2) \to \mathbb{K}$ is the 'bra' operator, i.e., such that $\langle x(y) := \langle x, y\rangle$.

We extend the action of the operators $\langle b$, $\langle c$, $\langle h$ to the algebra $\mathbb{K}[sl(2)^*]$ via relations (3.4), i.e. by means of the Leibnitz rule. Thus, the action

$$sl(2) \otimes \mathbb{K}[sl(2)^*] \to \mathbb{K}[sl(2)^*]$$

is well defined. It is clear that it is $SL(2)$-covariant. Otherwise stated, we have an $SL(2)$-covariant map

$$sl(2) \to \mathrm{Vect}(sl(2)^*) \tag{3.5}$$

different from that defined above via the Poisson bracket. We have associated a tangent (Poisson) vector field with any element from $sl(2)$, and now it is a partial derivative which is associated with such an element.

By using map (3.5) we associate a differential operator with any element from $U(sl(2))$. Thus, the Casimir element is related to the following operator

$$\Delta_{\mathbb{K}[sl(2)^*]} = \partial_b\partial_c + 2\partial_h^2 + \partial_c\partial_b.$$

It is a non-compact analog of the Laplace operator $\Delta_{\mathbb{K}[so(3)^*]}$. (More precisely, it is a multiple of the latter Laplacian written in the basis $\{b, h, c\}$.)

Similarly, the element $\mathrm{Id\,Cas} - \bar{\mathrm{e}}$ is related to the operator

$$\mathrm{Mw}_{\mathbb{K}[sl(2)^*]}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \begin{pmatrix}\Delta(\alpha)\\ \Delta(\beta)\\ \Delta(\gamma)\end{pmatrix} - \begin{pmatrix}\partial_b\\ \partial_h\\ \partial_c\end{pmatrix}(\partial_c, 2\partial_h, \partial_b)\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}, \qquad \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} \in \mathbb{K}[sl(2)^*]^{\oplus 3}. \tag{3.6}$$

Note that $(\partial_b\rho, \partial_h\rho, \partial_c\rho)^t \in \mathrm{Ker}(\mathrm{Mw}_{\mathbb{K}[\mathrm{sl}(2)^*]})$.

Now, we want to relate the vector fields $\langle b, \langle h, \langle c$ to the fields tangent to all hyperboloids. Observe that the following formula, similar to (2.2), holds:

$$r\partial_r = b\partial_b + h\partial_h + c\partial_c.$$

This entails the following relations, similar to (2.3),

$$\begin{aligned} &\langle b = \partial_c = \frac{hB - bH}{2r^2} + \frac{b}{r}\partial_r, \qquad \frac{1}{2}\langle h = \partial_h = \frac{bC - cB}{2r^2} + \frac{h}{2r}\partial_r,\\ &\langle c = \partial_b = \frac{cH - hC}{2r^2} + \frac{c}{r}\partial_r. \end{aligned} \tag{3.7}$$

Introduce the following notations

$$\mathcal{B} = \tfrac{1}{2}(hB - bH), \qquad \mathcal{H} = bC - cB, \qquad \mathcal{C} = \tfrac{1}{2}(cH - hC).$$

Thus, we obtain

$$\langle b = \frac{\mathcal{B}}{r^2} + \frac{b}{r}\partial_r, \qquad \langle h = \frac{\mathcal{H}}{r^2} + \frac{h}{r}\partial_r, \qquad \langle c = \frac{\mathcal{C}}{r^2} + \frac{c}{r}\partial_r. \tag{3.8}$$

Now, introduce a Laplace operator on a hyperboloid H_r^2 similarly to the previously described:

$$\Delta_{\mathbb{K}[\mathrm{H}_r^2]} = \frac{1}{r^4}\left(\mathcal{B}\mathcal{C} + \frac{\mathcal{H}^2}{2} + \mathcal{C}\mathcal{B}\right).$$

Also,

$$\Delta_{\mathbb{K}[\mathrm{H}_r^2]} = \frac{1}{r^2}\left(BC + \frac{H^2}{2} + CB\right).$$

According the pattern above we define the Maxwell operator on the hyperboloid H_r^2 as

$$\mathrm{Mw}_{\mathbb{K}[\mathrm{H}_r^2]}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = e\left(\begin{pmatrix}\Delta_{\mathbb{K}[\mathrm{H}_r^2]}(\alpha)\\ \Delta_{\mathbb{K}[\mathrm{H}_r^2]}(\beta)\\ \Delta_{\mathbb{K}[\mathrm{H}_r^2]}(\gamma)\end{pmatrix} - \frac{1}{r^4}\begin{pmatrix}\mathcal{C}\\ \frac{\mathcal{H}}{2}\\ \mathcal{B}\end{pmatrix}(\mathcal{B}, \mathcal{H}, \mathcal{C})\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}\right), \qquad \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} \in M\big(\mathrm{H}_r^2\big).$$

It is evident that the triples $\big(\mathcal{C}\rho, \frac{\mathcal{H}\rho}{2}, \mathcal{B}\rho\big)^t$ belong to the kernel $\mathrm{Ker}(\mathrm{Mw}_{\mathbb{K}[\mathrm{H}_r^2]})$.

Completing this section we want to emphasize that the Maxwell operators on the algebras $\mathbb{K}[\mathbb{R}^3]$ and $\mathbb{K}\big[\mathrm{H}_r^2\big]$ are $SL(2)$-invariant provided the matrices coming in the definition of these operators are endowed with a proper action of the group $SL(2)$. We exhibit a way to introduce such an action in the last section in a more general setting of quantum algebras.

4. Elements of analysis on the q-Minkowski space algebra

As has been mentioned above, the q-Minkowski space algebra is a particular case of the reflection equation algebra, which is defined as follows. Let $R : V^{\otimes 2} \to V^{\otimes 2}$ be a *braiding*, i.e., an invertible operator subject to the quantum Yang–Baxter equation

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R),$$

where V is an n-dimensional ($n \geqslant 2$) vector space over the field $\mathbb{K}$.

Let $L = \|l_i^j\|$ be a $n \times n$ matrix with entries l_i^j, $1 \leqslant i, j \leqslant n$. Then the relation

$$R(L \otimes I)R(L \otimes I) - (L \otimes 1)R(L \otimes I)R = 0 \tag{4.1}$$

is called the *reflection equation*. The algebra generated by the unity and the elements l_i^j subject to this system is called a *reflection equation algebra* (REA) and is denoted by $\mathcal{L}(R_q)$.

If, in addition, the braiding R is subject to the relation

$$(qI - R)(q^{-1}I + R) = 0, \qquad q \in \mathbb{K}$$

it is called a *Hecke symmetry*. If $q = 1$ it becomes an *involutive symmetry*.

Let $n = 2$ and R be the product of the image in the space $V^{\otimes 2}$ of the universal R-matrix of the QG $U_q(sl(2))$ and the usual flip. Then in the basis $\{x_1 \otimes x_1, x_1 \otimes x_2, x_2 \otimes x_1, x_2 \otimes x_2\}$ (where $\{x_1, x_2\}$ in an appropriate basis in V) the braiding R reads

$$R_q = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \tag{4.2}$$

It is easy to see that R is a Hecke symmetry. The parameter q is assumed to be generic.

In what follows the algebra $\mathcal{L}(R_q)$ corresponding to this Hecke symmetry is called the q-*Minkowski space algebra*. In this case it is also denoted $\mathbb{K}_q[\mathbb{R}^4]$. Also, we use the following notation for generators of this algebra

$$L = \begin{pmatrix} l_1^1 & l_1^2 \\ l_2^1 & l_2^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let us explicitly write down the system (4.1) with the Hecke symmetry (4.2):

$$\begin{array}{ll} qab - q^{-1}ba = 0 & q(bc - cb) = (q - q^{-1})a(d - a) \\ qca - q^{-1}ac = 0 & q(cd - dc) = (q - q^{-1})ca \\ ad - da = 0 & q(db - bd) = (q - q^{-1})ab. \end{array} \tag{4.3}$$

Now, rewrite the system (4.3) in the basis $\{l, h, b, c\}$ where $l = q^{-1}a + qd$, $h = a - d$:

$$\begin{array}{ll} q^2hb - bh = -(q - q^{-1})lb & bl = lb \\ q^2ch - hc = -(q - q^{-1})lc & cl = lc \\ (q^2 + 1)(bc - cb) + (q^2 - 1)h^2 = -(q - q^{-1})lh & hl = lh. \end{array} \tag{4.4}$$

Observe that the element l is central but, opposite to the classical case, it appears in the equations of the left column of the system (4.4). Also, we need the algebra $\mathbb{K}_q[\mathbb{R}^3] = \mathbb{K}_q[\mathbb{R}^4]/\langle l\rangle$ which is a braided analog of the coordinate algebra $\mathbb{K}[\mathbb{R}^3]$. It is generated by the three elements b, h, c subject to

$$q^2hb - bh = 0, \qquad q^2ch - hc = 0, \qquad (q^2 + 1)(bc - cb) + (q^2 - 1)h^2 = 0. \tag{4.5}$$

The generating spaces of this algebras $\mathrm{span}(a, b, c, d) = \mathrm{span}(b, h, c, l)$ and $\mathrm{span}(b, h, c)$ are respectively denoted $\mathcal{L}$ and $\mathcal{SL}$. Let us endow them with an action of the QG $U_q(sl(2))$ as follows.

Recall that the QG $U_q(sl(2))$ is generated by the unit and four generators K, K^{-1}, X, Y subject to the relations

$$KK^{-1} = 1, \qquad K^{\epsilon}X = q^{2\epsilon}XK^{\epsilon}$$
$$K^{\epsilon}Y = q^{-2\epsilon}YK^{\epsilon}, \qquad XY - YX = \frac{K - K^{-1}}{q - q^{-1}}, \qquad \epsilon \in \{-1, 1\}.$$

There exists a family of coproducts and corresponding antipodes which (together with the standard counit) endow this algebra with a Hopf structure. This family is parameterized by a continuous parameter θ assumed to be a real number (cf [DG]):

$$\Delta(K^{\epsilon}) = K^{\epsilon} \otimes K^{\epsilon}, \qquad \Delta(X) = X \otimes K^{\theta-1} + K^{\theta} \otimes X, \qquad \Delta(Y) = Y \otimes K^{-\theta} + K^{1-\theta} \otimes Y.$$

(In fact, all coproducts are equivalent, cf [DG].)

We define an action of the QG $U_q(sl(2))$ on the space $\mathcal{SL}$ as follows:

$$\begin{array}{lll} K^{\epsilon}(b) = q^{2\epsilon}b, & K^{\epsilon}(h) = h, & K^{\epsilon}(c) = q^{-2\epsilon}c, \\ X(b) = 0, & X(h) = -q^{\theta}2_q b, & X(c) = q^{1-\theta}h, \\ Y(b) = -q^{-\theta}h, & Y(h) = q^{\theta-1}2_q c, & Y(c) = 0. \end{array}$$

Hereafter, $n_q = \frac{q^n - q^{-n}}{q - q^{-1}}$. Moreover, we put $X(l) = Y(l) = 0$, $K^{\epsilon}(l) = l$. Thus, as a $U_q(sl(2))$ module, the space $\mathcal{L}$ is a direct sum of $\mathcal{SL}$ and $\mathbb{K}l$.

The reader can easily check that the structure of the algebra $\mathbb{K}_q[\mathbb{R}^3]$ is compatible with the action of the QG $U_q(sl(2))$ extended to higher powers of $\mathcal{SL}$ via the coproduct Δ. In order to do this it suffices to check that the system (4.5) is invariant with respect to the QG $U_q(sl(2))$.

Now, describe a regular way of introducing Casimir-like elements and covariant pairing into these spaces. In order to do this we need two special operators playing the role of the parity operators in super-spaces.

Let $R : V^{\otimes 2} \to V^{\otimes 2}$ be a braiding. It is called *skew-invertible* if there exists an operator $\Psi : V^{\otimes 2} \to V^{\otimes 2}$ such that

$$\mathrm{Tr}_2 R_{12}\Psi_{23} = P_{13},$$

where P_{13} is the flip transposing the first and third spaces in the product $V^{\otimes 3}$ and Tr_2 stands for the trace applied in the second space.

Fixing a basis $\{x_i\} \in V$ and representing the operators R and Ψ in the basis $\{x_i \otimes x_j\} \in V^{\otimes 2}$ by the matrices R_{ia}^{kb} and Ψ_{bj}^{al}, respectively, we rewrite this relation as follows:

$$\sum_{a,b=1}^{n} R_{ia}^{kb}\Psi_{bj}^{al} = \delta_i^l \delta_j^k.$$

For any skew-invertible Hecke symmetry the operators

$$B := \mathrm{Tr}_1\Psi\big(B_i^j = \Psi_{ai}^{aj}\big), \qquad C := \mathrm{Tr}_2\Psi\big(C_i^j = \Psi_{ia}^{ja}\big)$$

are well defined and the elements $\mathrm{Tr}_q L^k = \mathrm{Tr}\, CL^k$ are central in the algebra $\mathcal{L}(R_q)$ (cf [GPS1]).

The operator $L^k \to \mathrm{Tr}_q L^k$ is called the *quantum trace*. It can be extended by linearity to all polynomials of the matrix L. For the q-Minkowski space algebra $\mathbb{K}_q[\mathbb{R}^4]$ we have $C = \mathrm{diag}(q^{-3}, q^{-1})$. Hence, $l = q^2\,\mathrm{Tr}_q L$. Also, we consider the central element $q^2\,\mathrm{Tr}_q L^2$ in this algebra. In the basis $\{b, c, h, l\}$ its explicit form is

$$\mathrm{Cas}_{gl} = q^{-1}bc + qcb + \frac{1}{2_q}(h^2 + l^2). \tag{4.6}$$

Its image in the algebra $\mathbb{K}_q[\mathbb{R}^3]$ is

$$\mathrm{Cas}_{sl} = q^{-1}bc + \frac{1}{2_q}h^2 + qcb. \tag{4.7}$$

It is a central element in this algebra. So, we get braided analogs of the $gl(2)$ and $sl(2)$ Casimir elements, respectively. The quotient of the algebra $\mathbb{K}_q[\mathbb{R}^3]$ over the ideal generated by the element $\mathrm{Cas}_{sl} - r^2$ is called a *quantum (braided or q-)hyperboloid*.

Remark 6. Observe that Cas_{sl} is the unique quadratic element (up to a factor) in $\mathbb{K}_q[\mathbb{R}^3]$ which is $U_q(sl(2))$-invariant. As for the algebra $\mathbb{K}_q[\mathbb{R}^4]$, since the element l^2 is also $U_q(sl(2))$-invariant we have a family of such elements, namely, all linear combinations of Cas_{sl} and l^2.

We also need $U_q(sl(2))$-invariant pairings in the spaces $\mathcal{L} = \mathrm{span}(b, c, h, l)$ and $\mathcal{SL} = \mathrm{span}(b, c, h)$. For a general skew-invertible Hecke symmetry such a pairing on the space $\mathcal{L} = \mathrm{span}(l_i^j)$ is defined via the operator B. Namely, we put

$$\langle l_i^j, l_k^l\rangle = B_k^j \delta_i^l.$$

Also, an analog of the space $\mathcal{SL}$ can be defined in this case as the subspace of $\mathcal{L}$ which is the kernel of the linear map defined on the generators l_i^j by $l_i^j \to \delta_i^j$. Note that the space $\mathcal{L}$ can be identified with $\mathrm{End}(V)$ so that this map is nothing but a braided analog of the trace written in the basis $\{l_i^j\}$ (cf [GPS2]).

In the case of the q-Minkowski space algebra the pairing table is (we only exhibit terms with non-trivial pairing):

$$\begin{aligned} &\langle a, a\rangle = q^{-1}, \qquad \langle b, c\rangle = q^{-3}, \qquad \langle c, b\rangle = q^{-1}, \qquad \langle d, d\rangle = q^{-3} \\ &\text{hence} \qquad \langle h, h\rangle = q^{-2}2_q, \qquad \langle l, l\rangle = q^{-2}2_q. \end{aligned} \tag{4.8}$$

So, in contrast to the classical case, this pairing is not symmetric. Note that the restriction of this paring to the space $\mathcal{SL}$ is the unique (up to a factor) $U_q(sl(2))$-covariant pairing. However, we have a freedom for the pairing $\langle l, l\rangle$ on the space $\mathcal{L}$.

Our next goal is to define a braided analog of tangent vector fields. In order to do so, we endow the space $\mathcal{SL}$ with a braided analog of the Lie bracket $sl(2)$. Such analogs of Lie brackets $gl(n)$ and $sl(n)$ can be defined in the spaces $\mathcal{L}$ and $\mathcal{SL}$, respectively, for any skew-invertible Hecke symmetry R. We refer the reader to [GPS2] for their construction (also, see remark 9). But in the low-dimensional case in question we define such a bracket on the space $\mathcal{SL}$ by only using the fact that it is covariant.

Let us extend the action of the QG $U_q(sl(2))$ to the space $\mathcal{SL} \otimes \mathcal{SL}$, and decompose it in a direct sum of irreducible $U_q(sl(2))$ submodules $\mathcal{SL} \otimes \mathcal{SL} = V_0 \oplus V_1 \oplus V_2$ where the subscript stands for the spin. Then the operator

$$[\,,\,] : \mathcal{SL} \otimes \mathcal{SL} \to \mathcal{SL}$$

is a $U_q(sl(2))$ morphism iff it is trivial on the components V_0 and V_2, and it is an isomorphism between V_1 and $\mathcal{SL}$. By this condition the bracket is defined in the unique (up to a factor) way.

The multiplication table is as follows:

$$\begin{aligned} &[b, b] = 0, \quad [b, h] = -wb, \quad [b, c] = w\frac{q}{2_q}h, \quad [h, b] = wq^2 b, \\ &[h, h] = w(q^2 - 1), \quad [h, c] = -wc, \quad [c, b] = -w\frac{q}{2_q}h, \quad [c, h] = wq^2 c, \quad [c, c] = 0. \end{aligned}$$

Here w is an arbitrary factor. It can be fixed if we introduce the 'enveloping algebra' of this 'q-Lie algebra' by the relations

$$q^2hb - bh = \hbar b, \qquad (q^2 + 1)(bc - cb) + (q^2 - 1)h^2 = \hbar h, \qquad q^2ch - hc = \hbar c \tag{4.9}$$

and require the above bracket to define a representation of this enveloping algebra. Then $w = \hbar(q^4 - q^2 + 1)^{-1}$. We denote the space $\mathcal{SL}$ endowed with the above bracket $sl(2)_q$ and the algebra defined by the relations (4.9) $U(sl(2)_q)$.

Introduce q-analogs of the adjoint operators as follows $B_q = \mathrm{ad}\,(b)$, $H_q = \mathrm{ad}\,(h)$, $C_q = \mathrm{ad}\,(c)$ where the action ad is defined via the above bracket. These operators in the basis $\{b, h, c\}$ are

$$B_q = w\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & \frac{q}{2_q} \\ 0 & 0 & 0 \end{pmatrix} \quad H_q = w\begin{pmatrix} q^2 & 0 & 0 \\ 0 & q^2-1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad C_q = w\begin{pmatrix} 0 & 0 & 0 \\ -\frac{q}{2_q} & 0 & 0 \\ 0 & q^2 & 0 \end{pmatrix}. \tag{4.10}$$

Proposition 7. *The operators B_q, H_q, C_q are subject to*

$$q^{-1}bC_q + \frac{hH_q}{2_q} + qcB_q = 0. \tag{4.11}$$

Proof is straightforward.

Now, we want to extend the operator B_q, H_q, C_q to the algebra $\mathbb{K}_q[\mathbb{R}^3]$, preserving relation (4.11). It will be done by a slightly modified method suggested in [A].

Let V_k be the $U_q(sl(2))$ submodule of $\mathcal{SL}^{\otimes k}$ with the highest vector b^k, i.e.,

$$V_k = \mathrm{span}(b^k, Y(b^k), Y^2(b^k), \ldots, Y^{2k}(b^k)).$$

Note that $\dim V_k = 2k + 1$. There exists a $U_q(sl(2))$-covariant projector $P_k : \mathcal{SL}^{\otimes k} \to V_k$. It can be realized as a polynomial in

$$\mathrm{R}_{12} = \mathrm{R} \otimes I_{2k-1}, \qquad \mathrm{R}_{23} = I \otimes \mathrm{R} \otimes I_{2k-2}, \ldots, \qquad \mathrm{R}_{2k\,2k+1} = I_{2k-1} \otimes \mathrm{R},$$

where R is the product of the universal quantum R-matrix represented in the space $\mathcal{SL}^{\otimes 2}$ and the flip. A way of constructing such operators P_k is described in [OP]. Observe that the braiding R is of the Birman–Murakami–Wenzl type, and therefore the results of this paper can be applied. Then the extension of the operators B_q, H_q, $\mathcal{C}_q$ to the component V_k (denoted $B_q^{(k)}$, $H_q^{(k)}$, $\mathcal{C}_q^{(k)}$, respectively) are defined as follows:

$$B_q^{(k)} = \tau_k P_k(B_q \otimes I_{2k}), \qquad H_q^{(k)} = \tau_k P_k(H_q \otimes I_{2k}), \qquad C_q^{(k)} = \tau_k P_k(C_q \otimes I_{2k}),$$

where the factor τ_k can be found from the property that the operators $B_q^{(k)}$, $H_q^{(k)}$, $\mathcal{C}_q^{(k)}$ realize a representation of the algebra $U(sl(2)_q)$. Thus, the prolongation of the operators B_q, H_q, $\mathcal{C}_q$ is well defined on all components V_k.

Now, observe that for a generic q we have $\mathbb{K}_q[\mathbb{R}^3] \cong (\oplus V_k) \otimes Z$ where Z is the center of the algebra $\mathbb{K}_q[\mathbb{R}^3]$. By putting $B_q(v \otimes z) = B_q^k(v) \otimes z$ where $v \in V_k, z \in Z$ and analogously for H_q and C_q we define the operators B_q, H_q, C_q on the whole algebra $\mathbb{K}_q[\mathbb{R}^3]$. We call them *generating braided tangent vector fields*. By considering their combinations with coefficients from $\mathbb{K}_q[\mathbb{R}^3]$ we get, by definition, all *braided tangent vector fields*. This definition is motivated by the following.

Proposition 8. *The extended operators B_q, H_q, C_q are subject to the relation (4.11).*

Thus, the space of all braided tangent vector fields is treated as a left $\mathbb{K}_q[\mathbb{R}^3]$ module which is the quotient of the free module $\mathbb{K}_q[\mathbb{R}^3]^{\oplus 3}$ over the submodule generated by the lhs of (4.11).

Remark 9. Note that we have defined braided tangent vector fields without using any form of the Leibnitz rule (in other words, any coproduct in the algebra $\mathbb{K}_q[\mathbb{R}^3]$). Nevertheless, if R is a skew-invertible Hecke symmetry such a coproduct exists in the so-called modified REA which is a braided analog of the enveloping algebra $U(gl(n))$. This modified REA can be obtained from the REA $\mathcal{L}(R_q)$ by a shift (cf [GPS2] for details). It is possible to use this coproduct in order to define braided analogs of tangent vector fields in the space $\mathcal{SL}$ for $n > 2$. However, it is not clear whether these vector fields satisfy the relations looking like those of (4.11). Nevertheless, we want to point out the property of the mentioned coproduct to be a $U_q(sl(n))$ morphism. By using the construction of the paper [LS] it is also possible to get a series of representations of the algebra $U(sl(2)_q)$ close to those above. But this way of constructing the representations of this algebra does not give rise to braided tangent vector fields since the relation (4.11) is not satisfied.

5. Braided Maxwell operator on quantum algebras

In this section we define the Maxwell operator on the algebras $\mathbb{K}_q[\mathbb{R}^3]$, $\mathbb{K}_q\left[\mathrm{H}_r^2\right]$ and $\mathbb{K}_q[\mathbb{R}^4]$ following the classical pattern discussed above. First, we introduce braided analogs of the relations (3.7). In order to do this, we need to define the braided analogs of the operators $\mathcal{B}, \mathcal{H}, \mathcal{C}$. We put

$$\mathcal{B}_q = w^{-1}(q^2 hB_q - bH_q), \qquad \mathcal{H}_q = w^{-1}((q^2+1)(bC_q - cB_q) + (q^2-1)hH_q),$$
$$\mathcal{C}_q = w^{-1}(q^2 cH_q - hC_q).$$

Here w is a factor coming in (4.10). For the sake of convenience we put $w = 2_q$. Then for $q \to 1$ we retrieve the classical operators $\mathcal{B}, \mathcal{H}$ and $\mathcal{C}$, respectively.

In the second step we define the derivative ∂_r in the algebra $\mathbb{K}_q[\mathbb{R}^3]$ in the classical way, i.e., we assume the derivation in r to be subject to the Leibnitz rule and to act on the generators via the usual formula

$$\partial_r(b) = \frac{b}{r}, \qquad \partial_r(h) = \frac{h}{r}, \qquad \partial_r(c) = \frac{c}{r}. \tag{5.1}$$

Recall that r appears in the definition of a quantum hyperboloid, and observe that this way of introducing the derivative ∂_r is compatible with the defining relations of the algebras $\mathbb{K}_q[\mathbb{R}^3]$ and $\mathbb{K}_q[\mathbb{R}^4]$. Also, note that formulae (5.1) are invariant with respect to a renormalization $r \to ar, a \neq 0$. So, a way of normalizing the Casimir in the definition of the q-hyperboloid does not matter.

Observe the partial derivatives $\partial_b, \partial_h, \partial_c$ that on the space $\mathcal{SL}$ can be identified with 'bra' operators via the pairing (4.8). Namely, we have

$$\partial_b = q\langle c, \qquad \partial_h = \frac{q^2}{2_q}\langle h, \qquad \partial_c = q^3\langle b.$$

Now, we look for coefficients μ and ν such that the following holds on $\mathcal{SL}$:

$$\langle b = \mu\frac{\mathcal{B}_q}{r^2} + \nu\frac{b}{r}\partial_r, \qquad \langle h = \mu\frac{\mathcal{H}_q}{r^2} + \nu\frac{h}{r}\partial_r, \qquad \langle c = \mu\frac{\mathcal{C}_q}{r^2} + \nu\frac{c}{r}\partial_r. \tag{5.2}$$

Proposition 10. *These relations are valid with $\mu = q^{-4}$ and $\nu = q^{-2}$.*

Proof. We have to check that by applying the operator equalities (5.2) to any element from $\mathcal{SL}$ we get a correct result. For example, by applying the first equality to the element b we get a correct relation

$$0 = \langle b, b\rangle = \frac{q^2 hB_q(b) - bH_q(b)}{q^4 r^2} + \frac{b}{rq^2}\partial_r(b) = 0.$$

It is more difficult to check the three relations containing the pairings $\langle b, c\rangle$, $\langle c, b\rangle$ and $\langle h, h\rangle$. Let us check the first of them by leaving the others to the reader:

$$q^{-3} = \langle b, c\rangle = \frac{q^2 h B_q(c) - b H_q(c)}{q^4 r^2} + \frac{b}{rq^2}\partial_r(c) = \frac{1}{q^4 r^2}\left(q^2 h\left(\frac{qh}{2_q}\right) - b(-c)\right) + \frac{1}{q^2 r^2} bc$$
$$= \frac{h^2}{q 2_q r^2} + \frac{bc}{q^4 r^2} + \frac{1}{q^2 r^2}\left(cb + \frac{1-q^2}{1+q^2}h^2\right) = \frac{1}{q^3 r^2}\left(q^{-1}bc + \frac{h^2}{2_q} + qcb\right) = q^{-3}. \qquad \square$$

Note that the rhs of the relations (5.2) are well defined on the whole algebra $\mathbb{K}_q[\mathbb{R}^3]$ and extend the operators $\langle b$, $\langle h$, $\langle c$ to the algebra $\mathbb{K}_q[\mathbb{R}^3]$ via these relations. Finally, we arrive at the following definition of the Laplace operators in the algebras $\mathbb{K}_q[\mathbb{R}^3]$ and $\mathbb{K}_q[\mathrm{H}_r^2]$, respectively:

$$\Delta_{\mathbb{K}_q[\mathbb{R}^3]} = q^{-1}\langle b\langle c + \frac{1}{2_q}\langle h\langle h + q\langle c\langle b = q^{-5}\partial_c\partial_b + 2_q q^{-4}\partial_h\partial_h + q^{-3}\partial_b\partial_c,$$
$$\Delta_{\mathbb{K}_q[\mathrm{H}_r^2]} = \frac{1}{q^8 r^4}\left(q^{-1}\mathcal{B}_q\mathcal{C}_q + \frac{\mathcal{H}_q^2}{2_q} + q\mathcal{C}_q\mathcal{B}_q\right).$$

Now, we define the braided Maxwell operators on these algebras as follows:

$$\mathrm{Mw}_{\mathbb{K}_q[\mathbb{R}^3]}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \begin{pmatrix}\Delta_{\mathbb{K}_q[\mathbb{R}^3]}(\alpha)\\ \Delta_{\mathbb{K}_q[\mathbb{R}^3]}(\beta)\\ \Delta_{\mathbb{K}_q[\mathbb{R}^3]}(\gamma)\end{pmatrix} - \begin{pmatrix}\partial_b\\ \partial_h\\ \partial_c\end{pmatrix}(q^{-5}\partial_c, q^{-4}2_q\partial_h, q^{-3}\partial_b)\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}.$$

$$\mathrm{Mw}_{\mathbb{K}_q[\mathrm{H}_r^2]}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = e\left(\begin{pmatrix}\Delta_{\mathbb{K}_q[\mathrm{H}_r^2]}(\alpha)\\ \Delta_{\mathbb{K}_q[\mathrm{H}_r^2]}(\beta)\\ \Delta_{\mathbb{K}_q[\mathrm{H}_r^2]}(\gamma)\end{pmatrix} - \frac{1}{q^8 r^4}\begin{pmatrix}q^{-1}\mathcal{C}_q\\ \frac{\mathcal{H}_q}{2_q}\\ q\mathcal{B}_q\end{pmatrix}(\mathcal{B}_q, \mathcal{H}_q, \mathcal{C}_q)\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}\right)$$

where α, β, γ in these formulae belong to the algebras $\mathbb{K}_q[\mathbb{R}^3]$ and $\mathbb{K}_q[\mathrm{H}_r^2]$, respectively, $e = 1 - \bar{e}$, and

$$\bar{e} = \frac{1}{r^2}\begin{pmatrix}q^{-1}c\\ \frac{h}{2_q}\\ qb\end{pmatrix}\begin{pmatrix}b & h & c\end{pmatrix}.$$

Moreover, in the second formula we assume that the triples $(\alpha, \beta, \gamma)^t$ belong to the projective module $e\mathbb{K}_q[\mathrm{H}_r^2]^{\oplus 3}$ whereas in the first formula the columns $(\alpha, \beta, \gamma)^t$ form the free module $\mathbb{K}_q[\mathbb{R}^3]^{\oplus 3}$.

Let us justify this definition. The column $(\partial_b, \partial_h, \partial_c)^t$ in the first formula is universal (in the classical case it corresponds to the de Rham operator). The line $(q^{-5}\partial_c, q^{-4}2_q\partial_h, q^{-3}\partial_b)$ is chosen so that

$$\Delta_{\mathbb{K}_q[\mathbb{R}^3]} = (q^{-5}\partial_c, q^{-4}2_q\partial_h, q^{-3}\partial_b)\begin{pmatrix}\partial_b\\ \partial_h\\ \partial_c\end{pmatrix}.$$

The second formula can be obtained from the first one by disregarding the second summands in formulae (5.2). Now, we treat the module $e\mathbb{K}_q[\mathrm{H}_r^2]^{\oplus 3}$ as the space of braided differential forms $\Omega_q^1(\mathrm{H}_r^2)$ (for other modules in question it can be done in a similar way). The space $\Omega_q^1(\mathrm{H}_r^2)$ treated as a right $\mathbb{K}_q[\mathrm{H}_r^2]$ module is the quotient of the space of all braided differential forms $(db)\alpha + (dh)\beta + (dc)\gamma$ over the forms $(q^{-1}(db)c + \frac{(dh)h}{2_q} + q(dc)b)\rho$,

$\alpha, \beta, \gamma, \rho \in \mathbb{K}_q\big[\mathrm{H}_r^2\big]$. Note that we have defined the element $q^{-1}(db)c + \frac{(dh)h}{2_q} + q(dc)b$ by replacing the left factors in the Casimir Cas_{sl} by their 'differentials' without using either the Leibnitz rule or any transposing 'functions' and 'differentials'.

Similar to the classical pattern, we have the following.

Proposition 11.

(1) The triples $(\partial_b\rho, \partial_h\rho, \partial_c\rho)^t$ belong to $\mathrm{Ker}(\mathrm{Mw}_{\mathbb{K}_q[\mathbb{R}^3]})$ *provided the operator* $\Delta_{\mathbb{K}_q[\mathbb{R}^3]}$ *commutes with $\partial_b, \partial_h, \partial_c$.*

(2) The triples $\big(q^{-1}\mathcal{C}_q\rho, \frac{\mathcal{H}_q}{2_q}\rho, q\mathcal{B}_q\rho\big)^t$ belong to $\mathrm{Ker}\big(\mathrm{Mw}_{\mathbb{K}_q[\mathrm{H}_r^2]}\big)$ *provided*

$$e\left(\begin{pmatrix} q^{-1}\mathcal{C}_q \\ \frac{\mathcal{H}_q}{2_q} \\ q\mathcal{B}_q \end{pmatrix}\Delta_{\mathbb{K}_q[\mathrm{H}_r^2]} - \Delta_{\mathbb{K}_q[\mathrm{H}_r^2]}\begin{pmatrix} q^{-1}\mathcal{C}_q \\ \frac{\mathcal{H}_q}{2_q} \\ q\mathcal{B}_q \end{pmatrix}\right) = 0$$

on the algebra $\mathbb{K}_q\big[\mathrm{H}_r^2\big]$.

Now, we pass to the q-Minkowski space algebra $\mathbb{K}_q[\mathbb{R}^4]$ and discuss a way to convert the Casimir element (4.6) into an operator. Since $\mathrm{Cas}_{gl} = \mathrm{Cas}_{sl} + \frac{l^2}{2_q}$ and a method of assigning of an operator to the element Cas_{sl} is presented above, we only have to define the operator ∂_l on the algebra $\mathbb{K}_q[\mathbb{R}^4]$, keeping in mind that we have $\partial_l = \frac{q^2}{2_q}\langle l$ on the space $\mathcal{L}$. Since the element l is central, it is possible to define the extension of the derivative ∂_l via the usual Leibnitz rule. However, such a way is not compatible with the first column of the system (4.4). Nevertheless, conjecturally any element of the q-Minkowski space algebra $\mathbb{K}_q[\mathbb{R}^4]$ can be written in a completely q-symmetrized form as discussed in [GPS2], where this conjecture is proven for low-dimensional components. By assuming this conjecture to be true, we define the derivative ∂_l via the Leibnitz rule but only on elements presented in such a 'canonical' form.

Remark 12. In general, defining derivatives or vector fields it is often convenient to do this on basis elements. This prevents us from checking that the Leibnitz rule (if it is used in the definition of these operators) is compatible with defining relations. This idea is close to that of paper [G], where the construction of the Koszul complexes (similar to the de Rham complexes) uses 'R-symmetric' and 'R-skew-symmetric' algebras whose elements are realized in a '(skew)symmetrized' form. It is also similar to the above described construction of the operators B_q, H_q, C_q.

After having defined this derivative we can present the Laplace operator on the algebra in question in the following form

$$\Delta_{\mathbb{K}_q[\mathbb{R}^4]} = \Delta_{\mathbb{K}_q[\mathbb{R}^3]} + \frac{1}{2_q}\langle l\langle l = \Delta_{K_q[\mathbb{R}^3]} + \frac{2_q}{q^4}\partial_l\partial_l.$$

Also, define the Maxwell operator on this algebra

$$\mathrm{Mw}_{\mathbb{K}_q[\mathbb{R}^4]}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \begin{pmatrix}\Delta_{\mathbb{K}_q[\mathbb{R}^4]}(\alpha)\\ \Delta_{\mathbb{K}_q[\mathbb{R}^4]}(\beta)\\ \Delta_{\mathbb{K}_q[\mathbb{R}^4]}(\gamma)\\ \Delta_{\mathbb{K}_q[\mathbb{R}^4]}(\delta)\end{pmatrix} - \begin{pmatrix}\partial_b\\ \partial_h\\ \partial_c\\ \partial_l\end{pmatrix}\left(q^{-5}\partial_c, q^{-4}2_q\partial_h, q^{-3}\partial_b, \frac{2_q}{q^4}\partial_l\right)\begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{pmatrix}.$$

Here $\alpha, \beta, \gamma, \delta \in \mathbb{K}_q[\mathbb{R}^4]$.

Now, we introduce an action of the QG $U_q(sl(2))$ on the ingredients of the Maxwell operators in question so that these operators become $U_q(sl(2))$-invariant.

First, note that the QG acts on the operators B_q, H_q, C_q and on $\langle b, \langle h, \langle c$ in the same way as on the generators b, h, c. This means that the maps $b \to B_q, \ldots$ and $b \to \langle b, \ldots$ are $U_q(sl(2))$ morphisms. So, we restrict ourselves to the idempotent $\overline{e}$ and define the above action such that this idempotent becomes invariant. The reader can easily extend our method to other ingredients of the Maxwell operators.

Consider a representation of the algebra $U(sl(2)_q)$

$$\pi : b \to P^{-1}B_qP, \qquad h \to P^{-1}H_qP, \qquad c \to P^{-1}C_qP,$$

where P is an invertible numerical matrix: $P \in M_3(\mathbb{K})$. Define the action of the QG $U_q(sl(2))$ on the space $M_3(\mathbb{K})$ according to its action on the generators b, h, c:

$$K^{\epsilon}(P^{-1}B_qP) = q^{2\epsilon}P^{-1}B_qP, \ldots, Y(P^{-1}H_qP) = q^{\theta-1}2_qP^{-1}C_qP, Y(P^{-1}C_qP) = 0.$$

We extend this action to the whole space $M_3(\mathbb{K})$ by treating $M_3(\mathbb{K})$ as the image of the algebra $U(sl(2)_q)$. By construction, the representation $\pi : U(sl(2)_q) \to M_3(\mathbb{K})$ is a $U_q(sl(2))$ morphism.

Consider the element

$$w^{-1}\pi_2(\mathrm{Cas}_{sl}) = q^{-1}bP^{-1}\begin{pmatrix} 0 & 0 & 0 \\ -\frac{q}{2_q} & 0 & 0 \\ 0 & q^2 & 0 \end{pmatrix} P + \frac{h}{2_q}P^{-1}\begin{pmatrix} q^2 & 0 & 0 \\ 0 & q^2-1 & 0 \\ 0 & 0 & -1 \end{pmatrix} P$$
$$+qcP^{-1}\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & \frac{q}{2_q} \\ 0 & 0 & 0 \end{pmatrix} P = \frac{1}{2_q}P^{-1}\begin{pmatrix} q^2h & -2_qqc & 0 \\ -b & (q^2-1)h & q^2c \\ 0 & 2_qqb & -h \end{pmatrix} P.$$

Here π_2 means that we apply the representation π to the second factors of the *split* Casimir element, i.e. the Casimir element regarded as an element of $\mathcal{SL} \otimes \mathcal{SL}$.

Now, introduce the matrix $L = (w^{-1}\pi_2(\mathrm{Cas}_{sl}))^t$. Note the powers of this matrix are transposed to the 'powers' of the matrix $w^{-1}\pi_2(\mathrm{Cas}_{sl})$ where these 'powers' are defined via putting one matrix $w^{-1}\pi_2(\mathrm{Cas}_{sl})$ 'inside' of the other one. This procedure is described in detail in [GLS]. It is easy to see that these 'powers' are $U_q(sl(2))$-invariant. Also, note that the matrix L obeys a CH identity (cf [GLS, GS2]). Thus, introducing a new action of the QG $U_q(sl(2))$ on the elements $M \in M_3(\mathbb{K})$ as follows:

$$X \triangleright M = (X(M^t))^t, \qquad \forall X \in U_q(sl(2)),$$

where $X(M)$ is the above defined action, we see that the matrix L and all its powers are $U_q(sl(2))$-invariant.

Now, we claim that the idempotent $\overline{e}$ can be presented as a degree 2 polynomial of the matrix L for a special choice of the matrix P. Therefore, this idempotent is also invariant. We leave finding this polynomial and the corresponding matrix P to the reader.

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